Convective Model of Hartmann Flow

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Introduction

An analysis of the residence time distribution of a tracer in the fluid flowing through a continuous flow system is one of the important tools in the study of dynamic and dispersion characteristics of the system.[1–3] The residence time distribution is defined as the fraction of a tracer material introduced into a system at a dimensionless time $\theta = 0$ which appears at the outlet of the system between θ and $\theta + d\theta$.[1, 2] The tracer materials may be radioactive isotopes, electroconductive materials, dye stuffs, temperature variations, etc. If the injection of a tracer into the flow system is in the form of an impulse function such as a Dirac delta function, the response at the outlet of the system to the input commonly called the $\mathbf{C}(\theta)$ function is identical to the residence time distribution function (RTDF).[1, 2] Similarly, if the tracer is continuously applied to the inlet of the system in the form of a step input, the response at the outlet of the system is called the cumulative residence time distribution function or the $\mathbf{F}(\theta)$ function.[2] The $\mathbf{F}(\theta)$ function and $\mathbf{C}(\theta)$ function are related as[1, 2]

$$\mathbf{C}(\theta) = [d\mathbf{F}(\theta)/d\theta] \tag{1}$$

Since the residence time distribution function may be directly related to the degree of the tracer distribution either in longitudinal or transversal direction or both, it can be used to develop models to represent the dispersion and dynamic characteristics of flow systems.[1–3]

Convective Model: Velocity Profile Model

The equation of diffusion in the flow of an incompressible fluid through a parallel plate channel is

$$\frac{\partial C}{\partial t} + u(y)\frac{\partial C}{\partial x} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) \tag{2}$$

where D is the diffusivity, C the tracer concentration, t the time, and $u(y)$ is the x component fluid velocity, a function of y only. If the diffusion term can be neglected in comparison with the convective term, the steady-state velocity profile becomes the only factor governing the over-all or apparent tracer distribution.[3] The dynamic model of fluid flow corresponding to this specific situation is often called a convective (velocity profile) model.[3] For a two-dimensional parallel plate channel, the mathematical expression of $\mathbf{F}(\theta)$ function corresponding to the convective model is[4]

$$\mathbf{F}(\theta) = \frac{1}{\bar{u}L}\int_0^y u(y)dy \tag{3}$$

where y is a distance from a centerline of the channel in y direction, $\bar{u}$ is the mean fluid velocity, and L is the half depth of the channel.

Hartmann Flow

The velocity profile of steady magnetohydrodynamic flow in a parallel plate channel with a uniform transverse magnetic field H_0, which is called the Hartmann flow, is[5]

$$u(y) = \frac{PM}{u_e{}^2\sigma H_0{}^2}\left[\frac{\cosh M - \cosh(My/L)}{\sinh M}\right] \tag{4}$$

where M is the Hartmann number, $\mu_e H_0 L(\sigma/\rho\nu)^{1/2}$, P the pressure gradient in x direction, μ_e the magnetic permeability, and σ the electrical conductivity.

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Convective Model for Hartmann Flow

Rewriting Eq. (4) shows that

$$y = \frac{L}{M}\cosh^{-1}\left[\cosh M - \frac{\sinh M}{(PM/\mu_e{}^2\sigma H_0{}^2)}u(y)\right] \tag{5}$$

Since the dimensionless time is defined as a ratio of the time to mean residence time, i.e.,

$$\theta = t/\bar{t} = [\bar{u}/u(y)] \tag{6}$$

Eq. (5) can be expressed in a function of θ instead of $u(y)$, i.e.,

$$y = \frac{L}{M}\cosh^{-1}\left[\cosh M - \frac{\bar{u}\sinh M}{(PM/\mu_e{}^2\sigma H_0{}^2)}\frac{1}{\theta}\right] \tag{7}$$

In the range $[\cosh M - (\beta/\theta)]^2 > 1$, where

$$\beta = \frac{\bar{u}\sinh M}{(PM/\mu_e{}^2\sigma H_0{}^2)}$$

Eq. (7) can be expanded to a power series as shown below[6]:

$$y = \frac{L}{M}\left\{\ln 2\left(\cosh M - \frac{\beta}{\theta}\right) - \frac{1}{2}\cdot\frac{1}{2[\cosh M - (\beta/\theta)]^2} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{4[\cosh M - (\beta/\theta)]^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{6[\cosh M - (\beta/\theta)]^6} - \ldots\right\} \tag{8}$$

or

$$dy = \frac{L\beta}{M\theta^2}\left\{\frac{1}{[\cosh M - (\beta/\theta)]} + \frac{1}{2[\cosh M - (\beta/\theta)]^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{[\cosh M - (\beta/\theta)]^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{[\cosh M - (\beta/\theta)]^7} + \ldots\right\}d\theta = \frac{L\beta}{M\theta^2}\left\{\frac{1}{[\cosh M - (\beta/\theta)]} + \sum_{j=1}^{\infty}\frac{1\cdot 3\cdot 5\ldots(2j-1)}{2^j j![\cosh M - (\beta/\theta)]^{2j+1}}\right\}d\theta \tag{9}$$

Now the integration limits of Eq. (3) should be changed to a function of θ. The upper limit of the integration should correspond to θ. At the lower limit where $y = 0$, it can be seen from Eq. (7) that

$$\cosh M - \frac{\beta}{\theta} = 1 \quad \text{or} \quad \theta = \frac{\beta}{\cosh M - 1} \tag{10}$$

Equation (10) shows that it is equivalent to the shortest time the tracer can arrive at the outlet of the system because of the convection. But, as stated previously, the infinite series appearing in Eq. (9) converges only when the inequality $[\cosh M - (\beta/\theta)]^2 > 1$ or equivalently $[\cosh M - (\beta/\theta)] > 1$ holds. Solving this in terms of θ yields

$$\theta > \beta/(\cosh M - 1) \tag{11}$$

This indicates that the lower limit of the series convergence fortunately approaches to the lower limit of integration as indicated by Eq. (10). Substituting Eqs. (6) and (9) into Eq. (3), we have

$$\left.\begin{aligned}\mathbf{F}(\theta) &= \int_{\beta/(\cosh M - 1)}^{\theta}\frac{\beta}{M\theta^3}\times\left[\frac{1}{[\cosh M - (\beta/\theta)]} + \sum_{j=1}^{\infty}\frac{1\cdot 3\cdot 5\ldots(2j-1)}{2^j j![\cosh M - (\beta/\theta)]^{2j+1}}\right]d\theta \\ &\qquad\qquad \text{for } \theta > \frac{\beta}{\cosh M - 1} \\ &= 0 \qquad\qquad \text{for } \theta < \frac{\beta}{\cosh M - 1}\end{aligned}\right\} \tag{12}$$

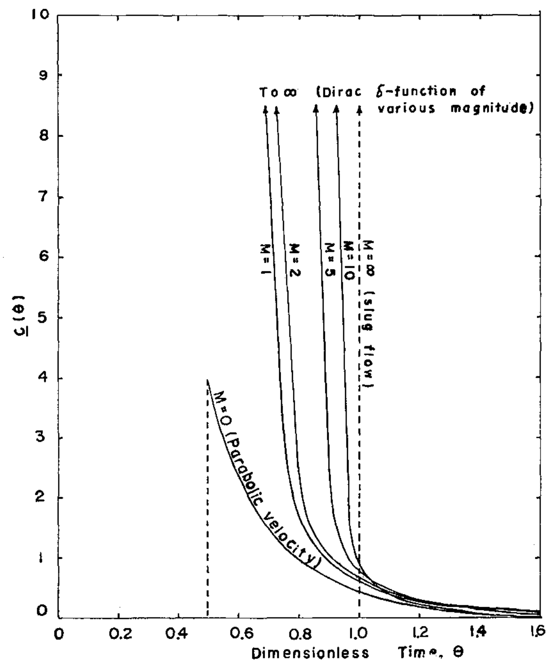

Fig. 1 C(θ) curves for Hartmann flow in a parallel plate channel with M as a parameter.

Since the $\mathbf{C}(\theta)$ function is the derivative of $\mathbf{F}(\theta)$ function with respect to θ, we obtain

$$\left.\begin{aligned} \mathrm{C}(\theta) &= \frac{\bar{u}\sinh M}{(PM/\mu_e^2\sigma H_0^2)\theta^3} \times \\ &\left\{\frac{1}{\{\cosh M - [\bar{u}\sinh M/(PM/\mu_e^2\sigma H_0^2)](1/\theta)\}} + \right. \\ &\left.\sum_{j=1}^{\infty}\frac{1\cdot3\cdot5\ldots(2j-1)}{2^j j!\{\cosh M - [\bar{u}\sinh M/(PM/\mu_e^2\sigma H_0^2)](1/\theta)\}^{2j+1}}\right\} \\ &\qquad \text{for } \theta > \frac{\bar{u}\sinh M}{(PM/\mu_e^2\sigma H_0^2)(\cosh M - 1)} \\ \mathrm{C}(\theta) &= 0 \text{ for } \theta < \frac{\bar{u}\sinh M}{(PM/\mu_e^2\sigma H_0^2)(\cosh M - 1)} \end{aligned}\right\} \tag{13}$$

The average velocity of the Hartmann flow, which is an easily measurable quantity, is[5]

$$\bar{u} = (P/\mu_e^2\sigma H_0^2)(M\cosh M - 1) \tag{14}$$

Substituting this expression into Eq. (13), it is simplified as

$$\left.\begin{aligned} \mathrm{C}(\theta) &= \frac{M\cosh M - \sinh M}{M^2\theta^3} \times \\ &\left\{\frac{1}{[\cosh M - (M\cosh M - \sinh M)/M\theta]} + \right. \\ &\left.\sum_{j=1}^{\infty}\frac{1\cdot3\cdot5\ldots(2j-1)}{2^j j![\cosh M - (M\cosh M - \sinh M)/M\theta]^{2j+1}}\right\} \\ &\qquad \text{for } \theta > \frac{M\cosh M - \sinh M}{M} \\ C\theta &= 0 \text{ for } \theta < \frac{M\cosh M - \sinh M}{M} \end{aligned}\right\} \tag{15}$$

The residence time distribution functions for various combinations of the dimensionless Hartmann number M can be calculated from Eq. (15). For illustration, several $\mathbf{C}(\theta)$ curves are plotted on Fig. 1.

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Application of Quasilinearization to Boundary-Layer Equations

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Introduction

THE analysis of boundary layers, which exhibit similarity, leads to ordinary differential equations with two-point asymptotic boundary conditions. That is, some boundary conditions are specified at the initial point or wall and others are specified as limits that must be approached at large values of the independent variable corresponding to the edge of the boundary layer.

Because of the inherent nonlinearity of the boundary-layer equations, as well as their considerable complexity for realistic formulations, e.g., compressibility effects, they are usually attacked by numerical integration using digital computers. Since initial conditions are required to start the integration, these must be assumed, based on physical intuition or otherwise. In general, the required boundary conditions at the second point are not satisfied. One method that has been used to find improved boundary conditions is that of making additional solutions with the guessed initial boundary conditions varied in turn followed by linear inverse interpolation (or extrapolation) on the boundary values attained at the second point.[1]

Another method, called successive substitutions or Picard's method, involves solving the differential equation for the highest derivative and substituting the result successively into the right-hand side starting with an approximation that satisfies the boundary conditions.[2] Both of these methods have been used to advantage; however; the first requirers a reasonably good approximation to the boundary conditions and the second to the functions throughout their range. Without sufficiently good starting values, the processes may diverge or converge extremely slowly. In addition, certain other problems arise which can be only named here, such as instability of solutions at large values of the independent variable or large changes of the solution at interior points for small errors in the boundary values.

The purpose of this note is to show by means of a simple example that the quasilinearization method gives promise of producing rapid convergence to solutions of boundary-layer problems from uninspired initial guesses.

Discussion of the Method

Quasilinearization may be viewed as an extension of Newton's method for the solution of algebraic equations to solution of differential equations.[2] Consider the vector equation

$$d\mathbf{X}/dt = \mathbf{g}(\mathbf{X}) \tag{1}$$

where $\mathbf{X}$ is a vector composed of the n-dependant variables in

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